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POWER SERIES METHODS I -  
ORDINARY DIFFERENTIAL EQUATIONS

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POWER SERIES METHODS I - ORDINARY

DIFFERENTIAL EQUATIONS

Robert D. Small

Technical Summary Report #1923  
February 1979

ABSTRACT

A method is presented for constructing finite difference schemes of high accuracy for ordinary differential equations. Since it is based on power series representation of the solution it is applicable to partial differential equations as well. The method is demonstrated on the general nonlinear first order ordinary differential equation and the truncation errors are analysed. A-stability of the method is analysed and it is found to be stable in this sense to arbitrarily high order. Finally a nonlinear example is presented.

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## SIGNIFICANCE AND EXPLANATION

The author proposes a method for solving partial differential equations numerically. The present paper introduces the method by applying it to ordinary differential equations. The distinctive feature of this method is that it may be applied in more and more accurate versions in a simple way, and in this respect it is more easily used than some classical methods such as the Runge-Kutta method. When applied to partial differential equations it has no equivalent since it is of high accuracy in both variables. The method is based on an old classical technique for solving ordinary differential equations where power series in  $x$  are substituted directly into the equation.

One difficulty with numerical methods is that errors introduced in the course of generating solutions may grow so that results may become nonsensical. When this happens the method is said to be unstable. The proposed method is analysed with this in mind and is shown to be stable to all orders in a particular sense. The term A-stable is due to Dalquist, and indicates that a computed solution decreases when the exact one does for a test equation that he proposed.

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POWER SERIES METHODS I - ORDINARY  
DIFFERENTIAL EQUATIONS

Robert D. Small

Introduction

The purpose of the series of papers of which this is the first is to present a stable finite difference scheme for partial differential equations that is of high accuracy in all variables. Traditionally, beginning with Euler's method, finite difference schemes have been constructed by approximating the derivatives that appear in an equation by linear combinations of function values at equally spaced points along a grid. When placed into the equations the result is a difference equation in the function values at grid points. A major difficulty in taking a high order approximation for each term in the equation is that the resulting difference equation is very often unstable. One way of overcoming this problem is to approximate, to high order, the solution rather than the terms of the equation. A great variety of methods known as finite element methods fall in this category. Examples are Galerkin methods that fit solution surfaces over the cells into which the domain is partitioned according to some criterion that minimizes the error over each cell and collocation methods that minimize or annihilate the error at specific points in the cells. These methods have achieved considerable success at producing accurate and stable methods of approximating solutions to partial differential equations but at the expense of complication in their use and analysis especially when high accuracy is desired in all variables. The author proposes an alternative method that is a mild extension of the power series method that is customarily used on the ordinary differential equations whose solutions are special functions. In this respect it resembles Runge-Kutta schemes. In fact when Runge-Kutta schemes are used for one of the variables [1] in parabolic equations it is the use of implicit schemes that promotes stability just as it is in the power series method we present.

The essence of the method is to partition the domain into rectangular cells and to fit solution surfaces over each cell by means of power series. The differential equation and boundary conditions or continuity conditions then fix all the coefficients of the power series. The point of expansion of each series is chosen in order that the resulting scheme be stable when the series are truncated and this is usually the center of the cell. The accuracy of the scheme depends on the number of terms of the series that are retained at truncation and on the size of the cells.

The purpose of this paper is to present a stable form of the power series method for ordinary differential equations that generates finite difference schemes of high order. It has stability properties similar to the implicit Runge-Kutta methods and accuracy that compares favorably with those methods. Its advantage, however, is that it may be straightforwardly applied to each variable in partial differential equations. The present paper lays the groundwork for the derivation of schemes and the stability and error analysis in partial differential equations that will be dealt with in a later paper. To that end the method is applied to first order nonlinear ordinary differential equations and truncation error is investigated. Then the stability of the method in the sense of Dalquist is studied. This detailed analysis turns out to be of great importance when dealing with the heat equation in the paper to follow. Finally a nonlinear example is computed.

# 1. The Method

We demonstrate the method on the first order nonlinear equation

$$(1) \quad \frac{dy}{dx} = f(x, y)$$

where  $f$  is analytic in both  $x$  and  $y$  at some point  $(\bar{x}, a_0)$ . We take the solution  $y(x)$  of the form

$$(2) \quad y = \sum_{n=0}^{\infty} \frac{a_n (x - \bar{x})^n}{n!},$$

and, substituting into (1) we obtain the following hierarchy of relations by equating like powers of  $x - \bar{x}$ .

$$\begin{aligned} a_1 &= f^{(0,0)} \\ a_2 &= f^{(1,0)} + a_1 f^{(0,1)} \\ a_3 &= a_2 f^{(0,1)} + f^{(2,0)} + 2a_1 f^{(1,1)} + a_1^2 f^{(0,2)} \\ (3) \quad a_4 &= a_3 f^{(0,1)} + 3a_2 f^{(1,1)} + 3a_1 a_2 f^{(0,2)} + f^{(3,0)} + 3a_1 f^{(2,1)} + 3a_1^2 f^{(1,2)} + a_1^3 f^{(0,3)} \\ &\vdots \end{aligned}$$

We have used the notation

$$f^{(n,m)} = \left. \frac{\partial^{n+m} f(x,y)}{\partial x^n \partial y^m} \right|_{\substack{x = \bar{x} \\ y = a_0}}.$$

This set of equations evaluates the  $a_n$  sequentially and hence the solution (2) now depends on the unknowns  $a_0$  and  $\bar{x}$ . The analyticity of  $f$  is sufficient to define all the  $a_n$  from (3) and hence the function  $y(x)$  is analytic at  $\bar{x}$ .

We may assume that the solution is known at some point  $x_0$  and we wish to find it at the point  $x_0 + h$ . If we choose  $\bar{x} = x_0$  then  $a_0$  is known immediately, starting sequence (3) and then (2) may be evaluated at  $x_0 + h$ . This is known as an explicit method since the solution is calculated by successive substitutions and it is precisely the power series method used to evaluate ordinary differential equations expanded about analytic



points. As we shall see, when (2) is truncated at some order of accuracy, this procedure can be unstable in the sense of Dalquist [2]. Our prime motivation, however, is to produce a method that is stable when (1) is a partial differential equation due to the dependence of  $f$  on further independent variables and operators in these variables. In this case the explicit method is almost invariably unstable in the higher approximations.

To alleviate the instability we place the point of expansion,  $\bar{x}$ , not at the point where the solution is known, but elsewhere in the interval  $(x_0, x_0 + h)$ . Then  $y(x_0)$  and  $y(x_0 + h)$  both depend in a complicated way on  $a_0$  which is now a parameter. The solution  $y(x_0 + h)$  becomes known implicitly in terms of  $y(x_0)$ . If we set

$$\bar{x} = x_0 + \lambda h$$

we have the relations

$$(4) \quad \begin{aligned} y(x_0) &= \sum_{n=0}^{\infty} \frac{a_n (-\lambda h)^n}{n!} \\ y(x_0 + h) &= \sum_{n=0}^{\infty} \frac{a_n ((1 - \lambda)h)^n}{n!} \end{aligned}$$

where the  $a_n$  are given by (3). These equations determine  $y(x_0 + h)$  when  $a_0$  is eliminated. The implicit solution (4) may now be approximated by truncating the series at the finite value  $n = N$ , and the solution is recovered by truncating (2) at the same place, finally giving the set

$$(5) \quad \begin{aligned} y(x_0) &= \sum_{n=0}^N \frac{\bar{a}_n (-\lambda h)^n}{n!} \\ y(x_0 + h) &= \sum_{n=0}^N \frac{\bar{a}_n ((1 - \lambda)h)^n}{n!} \\ y(x) &= \sum_{n=0}^N \frac{\bar{a}_n (x - \bar{x})^n}{n!} \end{aligned}$$

where the  $\bar{a}_n$  satisfy (3) for  $n = 1, 2, \dots, N$ . Having produced a known value of the function at  $x_0 + h$  we can repeat the procedure and determine the solution at larger and larger values of  $x$  by shifting all  $x$  values in (3) and (5) by  $h$  at each step.



It happens that choosing  $\bar{x}$  to be the mid-point of the interval gives a stable scheme for the linear case and we analyse this in section 3. For the present, however, we investigate the restrictions on  $f(x,y)$  that lead to a workable scheme, and the accuracy that results from a choice of  $N$ .

## 2. Truncation Error and Consistency

It has been noted that  $f(x,y)$  must be analytic in  $x$  and  $y$  at all points  $(\bar{x}, \bar{a}_0)$  found in the course of the computation in order to define the  $a_n$ . We consider the  $a_n$  as functions of  $a_0$  from (3) and regard  $a_0$  as dependent on  $h$  and  $y(x_0)$  when (4) is enforced. Then to have the  $a_n$  bounded as  $h \rightarrow 0$  we require that  $f(x,y)$  be analytic in  $x$  and  $y$  for all points  $(x_0 + \bar{h}, a_0)$  in some range  $0 < \bar{h} \leq h$ . We can now compare the  $\bar{a}_n$  to the  $a_n$ . By means of (4) and (5) we have

$$(6) \quad \sum_{n=0}^N \frac{\bar{a}_n (-\lambda h)^n}{n!} = \sum_{n=0}^{\infty} \frac{a_n (-\lambda h)^n}{n!}.$$

Since the  $\bar{a}_n$  are established by (3) we can write  $\bar{a}_n = a_n(\bar{a}_0)$  so that (6) gives  $\bar{a}_0$  in terms of  $a_0$ . We can notice an approximation to the solution for small  $h$  by setting

$$\bar{a}_0 = a_0 + \gamma(h),$$

which gives

$$\bar{a}_n = a_n(a_0) + \gamma \frac{da_n(a_0)}{da_0}.$$

Placing these into (6) the leading order solution for  $\gamma$  is

$$(7) \quad \gamma = \frac{a_{N+1} (-\lambda h)^{N+1}}{(N+1)!}.$$

We then obtain the result that the  $\bar{a}_n$  differ from the  $a_n$  by terms of  $O(h^{N+1})$  and the higher values of  $N$  introduce factorials that help out as well. There is the difficulty that (6) is a nonlinear equation and the noticed solution for fixed  $h$  may not be unique. It is unique as  $h \rightarrow 0$ , however, and this may restrict the size of  $h$  if one cannot find a method for identifying spurious roots of (5).

We now show that the difference scheme (5) is consistent with the differential equation (1) as  $h \rightarrow 0$ . Displaying the result of substituting (5) into each side of (1) we have

$$\frac{dy}{dx} = \sum_{n=0}^{N-1} \frac{\bar{a}_{n+1} (x - \bar{x})^n}{n!}$$

$$(8) \quad f(x,y) = f + (f^{(1,0)} + \bar{a}_1 f^{(0,1)}) (x - \bar{x}) + \dots$$

$$\dots + (\text{Nth expression in equations (3)}) (x - \bar{x})^N$$

+ higher order terms .

The arguments of  $f$  and its derivatives are  $(\bar{x}, \bar{a}_0)$ . The "higher order terms" are the first to feel the truncation of (5), the preceding terms containing identical expressions to the right sides of equations (3). Comparing the two expressions in (8) the first discrepancy is that the term  $\frac{\bar{a}_N (x - \bar{x})^N}{N!}$  is present in  $f(x,y)$  but absent in  $\frac{dy}{dx}$  and this gives an error of  $O(h^N)$ . For  $N \geq 1$  the error vanishes as  $h \rightarrow 0$  and we have a difference scheme that is consistent with the differential equation.

The error in the solution is found by comparing the last equation of (5) with (2). The difference of the expressions is of  $O(h^{N+1})$ .

We have found that  $f(x,y)$  must be analytic in the range of  $x$  between the point where the solution is known,  $x_0$ , and the point of expansion,  $\bar{x}$ , and for all solution values  $y$  that could result there. Solving backwards from any solution found for  $x$  to the right of the point of expansion one uses the same argument to indicate that  $f(x,y)$  should be analytic for that interval as well. When  $f(x,y)$  has a point of non-analyticity in  $x$ , that point may be used as an end point of an interval and then the severity of the singularity determines whether the series (2) converges. For example if  $f(x,y)$  changes formulas as it does at the interface of regions with different properties, the solution is continued without difficulty by making the point of interface separate two intervals.



### 3. A-Stability of the Method

We now investigate the stability of the proposed method on a model problem proposed by Dalquist [2] for testing numerical methods. A method is said to be A-stable if the numerical solution to the equation

$$(9) \quad \frac{dy}{dx} = qy$$

decays for all initial conditions when  $\text{Re } q < 0$ . Hamming [3] correctly criticizes indiscriminant use of this definition of stability since errors can still swamp the solution if they decay more slowly than the solution or solutions can dominate errors if they grow faster than the errors. We prefer Dalquist's definition over others, however, for its later significance to partial differential equations.

Since (9) is autonomous we can take the solution of the form

$$y = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!}$$

and substitute into the equation to obtain

$$a_{n+1} = qa_n, \quad n = 0, 1, \dots, \infty.$$

The solution to this difference equation is  $a_n = q^n a$  and hence we obtain

$$(10) \quad y = \sum_{n=0}^{\infty} \frac{q^n x^n}{n!} a$$

where  $a$  is a constant yet to be determined. If the solution is known at some point  $x_0$  and we wish to find it at  $x_0 + h$ , then we choose a new origin within this interval at a distance  $\lambda h$  from  $x_0$ . Thus by a translation of the original co-ordinates we may take  $x_0 = -\lambda h$  and  $x_0 + h = (1 - \lambda)h$ . Placing these values of  $x$  into (10) and eliminating  $a$  we obtain

$$y(x_0 + h) = Ry(x_0)$$

where



$$(11) \quad R = \sum_{n=0}^{\infty} \frac{((1-\lambda)qh)^n}{n!} \bigg/ \sum_{n=0}^{\infty} \frac{(-\lambda qh)^n}{n!}.$$

This exact answer is approximated by truncating each series at the term  $n = N$ . The method is A-stable if  $|\bar{R}| < 1$ , where the ratio of finite sums,  $\bar{R}$ , is given by

$$(12) \quad \bar{R} = \sum_{n=0}^N \frac{((1-\lambda)qh)^n}{n!} \bigg/ \sum_{n=0}^N \frac{(-\lambda qh)^n}{n!}.$$

We set  $qh = -\alpha + i\beta$  in (12) with  $\alpha$  positive and write  $\bar{R}$  as

$$(13) \quad \bar{R} = \frac{\sum_{n=0}^N P_n (1-\lambda)^n/n! + i \sum_{n=0}^N Q_n (1-\lambda)^n/n!}{\sum_{n=0}^N P_n (-\lambda)^n/n! + i \sum_{n=0}^N Q_n (-\lambda)^n/n!}$$

where

$$P_n = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2j} (-1)^{j+n} \alpha^{n-2j} \beta^{2j}$$

and

$$Q_n = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2j+1} (-1)^{n+j+1} \alpha^{n-2j-1} \beta^{2j+1}.$$

Then  $|\bar{R}| < 1$  may be written

$$\left( \sum_{n=0}^N P_n (1-\lambda)^n/n! \right)^2 + \left( \sum_{n=0}^N Q_n (1-\lambda)^n/n! \right)^2 < \left( \sum_{n=0}^N P_n (-\lambda)^n/n! \right)^2 + \left( \sum_{n=0}^N Q_n (-\lambda)^n/n! \right)^2.$$

Simplification of this inequality leads to the stability condition

$$(14) \quad \sum_{n=0}^N \sum_{j=0}^n S_{nj} f_n(\lambda) + \sum_{n=N+1}^{2N} \sum_{j=n-N}^N S_{nj} f_n(\lambda) > 0$$

where

$$S_{nj} = \frac{P_j P_{n-j} + Q_j Q_{n-j}}{j!(n-j)!}$$

and

$$f_n(\lambda) = (-\lambda)^n - (1-\lambda)^n.$$

The first term of (14) simplifies further giving

$$(15) \quad \sum_{n=0}^N \frac{(-1)^n 2^n \alpha^n}{n!} f_n(\lambda) + \sum_{n=N+1}^{2N} \sum_{j=n-N}^N s_{nj} f_n(\lambda) > 0$$

for stability. We consider (15) as a condition on  $\lambda$  for given  $\alpha$ ,  $\beta$  and  $N$ . First we note two facts about the  $f_n(\lambda)$ , both easily proved by induction.

$$f_{2n} \begin{cases} < 0 & \text{for } \lambda < \frac{1}{2} \\ = 0 & \text{for } \lambda = \frac{1}{2} \\ > 0 & \text{for } \lambda > \frac{1}{2} \end{cases} \quad n = 1, 2, 3, \dots, \infty$$

$$f_{2n+1} < 0 \quad \text{for all } \lambda, \quad n = 0, 1, 2, \dots, \infty$$

Using both these facts we see that the first term of (15) is a sum of positive terms if  $\lambda > \frac{1}{2}$ . The second term must be evaluated by brute force. We display the first few orders of (15) for illustration.

$$N = 1 \quad 2\alpha + (\alpha^2 + \beta^2)f_2 > 0$$

$$N = 2 \quad 2\alpha + 2\alpha^2 f_2 - (\alpha^3 + \alpha\beta^2)f_3 + \frac{1}{4}(\alpha^4 + 2\alpha^2\beta^2 + \beta^4)f_4 > 0$$

$$N = 3 \quad 2\alpha + 2\alpha^2 f_2 - \frac{4}{3}\alpha^3 f_3 + \frac{1}{12}(7\alpha^4 + 6\alpha^2\beta^2 - \beta^4)f_4 - \frac{1}{6}(\alpha^5 + 2\alpha^3\beta^2 + \alpha\beta^4)f_5 \\ + \frac{1}{36}(\alpha^6 + 3\alpha^4\beta^2 + 3\alpha^2\beta^4 + \beta^6)f_6 > 0$$

We recall that  $\alpha > 0$  but  $\beta$  may have either sign. Then the inequalities for  $N = 1, 2$ , are satisfied if  $\lambda \geq \frac{1}{2}$ . For  $N = 3$  the only term that is not positive for  $\lambda \geq \frac{1}{2}$  is  $-\frac{1}{12}\beta^4 f_4$ . If the scheme is to be independent of  $\alpha$  and  $\beta$  then we must choose  $\lambda = \frac{1}{2}$  to suppress the offensive term. If the scheme may be adapted to the given values of  $\alpha$  and  $\beta$  then further stable values of  $\lambda$  can be found since sufficiently large values make the last term of the inequality dominate. These are unreasonable choices, however, since  $\lambda = \frac{1}{2}$  is a more accurate scheme.

For higher values of  $N$  we encounter more and more terms whose signs work against the inequality. As in the case  $N = 3$ , we take  $\lambda = \frac{1}{2}$  to suppress the terms with even

subscript in  $f$ . It turns out that for  $N = 4$  this is sufficient for that inequality to hold unconditionally, but for  $N \geq 5$  there are always points  $(\alpha, \beta)$  that violate (15). To locate these points we set  $|\bar{R}| = 1$  and  $\lambda = \frac{1}{2}$  in (12) and search for roots numerically. The solutions are curves in the complex plane of the variable  $qh$  that separate the regions  $|\bar{R}| > 1$  and  $|\bar{R}| < 1$ . It was found that writing  $qh$  in polar co-ordinates gave the simplest form for programming.

The results are plotted in Figure 1. The regions of instability are the interiors of the closed curves located in the left half plane corresponding to the  $N$  of interest. The several chains of curves have been shifted vertically since they actually overlap at their bases. To obtain an A-stable scheme for equation (9) for a given accuracy  $N$ , we must choose  $h$  in such a way that  $qh$  does not fall in one of the undesirable regions when  $\text{Re } q < 0$ . This is clearly an easy matter since the regions are relatively small and sparse. Of course A-stability, as we have noted before, is just a criterion that indicates whether the computed solution decays when the exact one does and Figure 1 similarly indicates that if  $qh$  avoids the interiors of curves in the right half plane, computed solutions will grow only when exact ones do. Figure 1 and equation (16), to be derived, will have considerable importance when dealing with partial differential equations.

To study these curves further we rewrite (12) in yet another way. Setting  $\lambda = \frac{1}{2}$  in (12) we collect powers of  $\frac{qh}{2}$  to obtain

$$\sum_{n=0}^{\lfloor \frac{N}{2} \rfloor} \frac{\left(\frac{qh}{2}\right)^{2n}}{(2n)!} = \frac{\bar{R} + 1}{\bar{R} - 1} \sum_{n=0}^{\lfloor \frac{N-1}{2} \rfloor} \frac{\left(\frac{qh}{2}\right)^{2n+1}}{(2n+1)!}.$$

When  $|\bar{R}| = 1$  the factor  $(\bar{R} + 1)/(\bar{R} - 1)$  is purely imaginary and the circle in the  $\bar{R}$  plane maps into the entire imaginary axis. Thus we set

$$ic = \frac{\bar{R} + 1}{\bar{R} - 1}$$

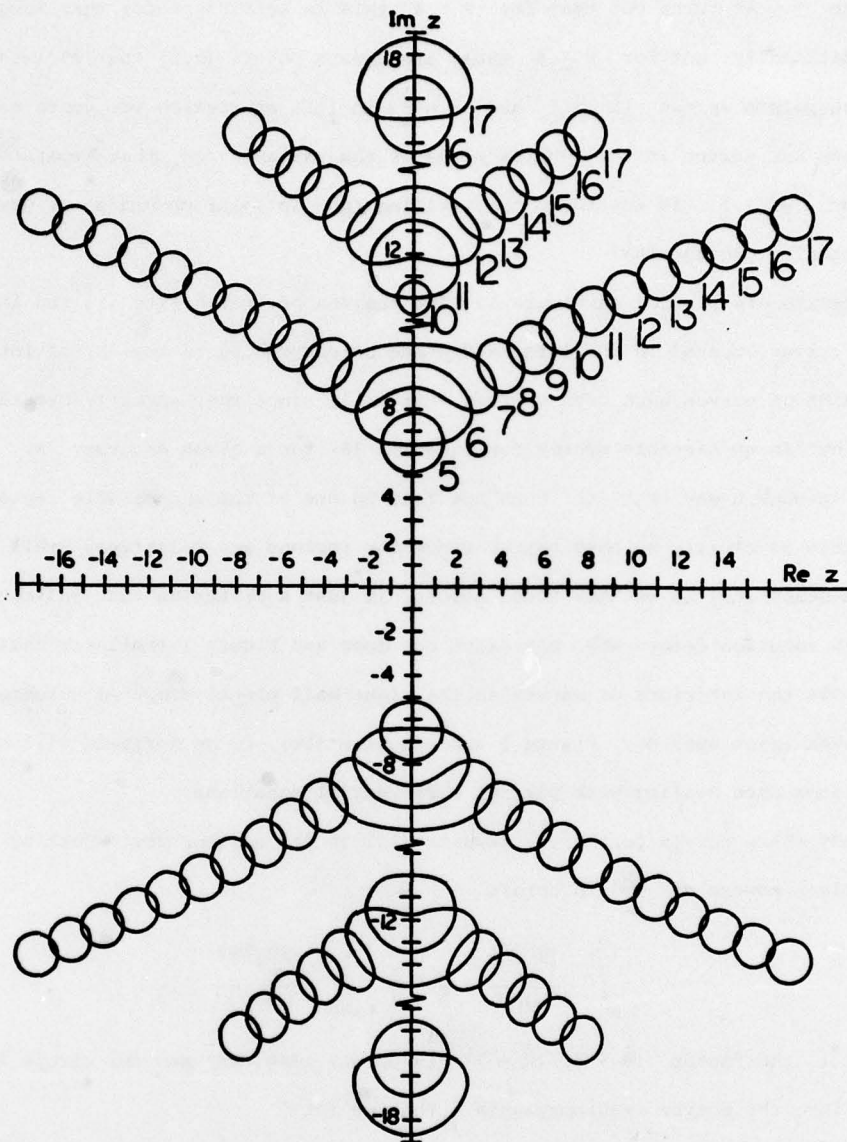
and take in addition,

$$z = \frac{qh}{2}, \quad N_1 = \lfloor \frac{N}{2} \rfloor, \quad N_2 = \lfloor \frac{N-1}{2} \rfloor,$$

to obtain



Figure 1. Regions of Instability



These curves are the loci of roots of the equation

$$\sum_{n=0}^{\lfloor \frac{N}{2} \rfloor} \frac{z^{2n}}{(2n)!} = ic \sum_{n=0}^{\lfloor \frac{N-1}{2} \rfloor} \frac{z^{2n+1}}{(2n+1)!}$$

for real parameter  $c$ . The large numbers indicate the value of  $N$  for which the curve applies. The imaginary axis is a locus for all  $N$ . For comparison with the text,  $z = \frac{qh}{2}$ .



$$(16) \quad \sum_{n=0}^{N_1} \frac{z^{2n}}{(2n)!} = ic \sum_{n=0}^{N_2} \frac{z^{2n+1}}{(2n+1)!} .$$

The curves of Figure 1 are the roots of the  $N$ th degree equation (16) parameterized by real  $c$ . The imaginary axis is also a locus of roots of (16) for all  $N$  since any imaginary  $z$  corresponds to a real  $c$ . Inspection of Figure 1 indicates that whenever a curve has a vertical tangent it intersects a curve of neighboring order. This fact will be proved from (16) and it then serves to indicate that the chains of curves are infinite in extent.

Since  $c$  parameterizes the curve, a tangent vector to the curve is given by

$$\frac{dz}{dc} = \frac{i \sum_{n=0}^{N_2} \frac{z^{2n+1}}{(2n+1)!}}{\sum_{n=0}^{N_1-1} \frac{z^{2n+1}}{(2n+1)!} - ic \sum_{n=0}^{N_2} \frac{z^{2n}}{(2n)!}} .$$

We write  $z = re^{i\theta}$  and introduce the notation

$$c_1^j = \sum_{n=0}^j \frac{r^{2n} \cos 2n\theta}{(2n)!} , \quad s_1^j = \sum_{n=0}^j \frac{r^{2n} \sin 2n\theta}{(2n)!}$$

$$c_2^j = \sum_{n=0}^j \frac{r^{2n+1} \cos (2n+1)\theta}{(2n+1)!} , \quad s_2^j = \sum_{n=0}^j \frac{r^{2n+1} \sin (2n+1)\theta}{(2n+1)!} .$$

Then the derivative, written as real and imaginary parts becomes

$$(17) \quad \frac{dz}{dc} = \frac{-s_2^{N_2} \left( c_2^{N_1-1} + cs_1^{N_2} \right) + c_2^{N_2} \left( s_2^{N_1-1} - cc_1^{N_2} \right) + i[1]}{\left( c_2^{N_1-1} + cs_1^{N_2} \right)^2 + \left( s_2^{N_1-1} - cc_1^{N_2} \right)^2}$$

$$\text{where } [1] = c_2^{N_2} \left( c_2^{N_1-1} + cs_1^{N_2} \right) + s_2^{N_2} \left( s_2^{N_1-1} - cc_1^{N_2} \right) .$$

Written in new notation, (16) is

$$(18) \quad c_1^{N_1} = -cs_2^{N_2}$$

$$s_1^{N_1} = cc_2^{N_2} ,$$

and with  $c$  eliminated, the curve of order  $N$  is

$$(19) \quad c_1^{N_1} c_2^{N_2} + s_1^{N_1} s_2^{N_2} = 0.$$

Similarly the curve of order  $N - 1$  is

$$(20) \quad c_1^{N_2} c_2^{N_3} + s_1^{N_2} s_2^{N_3} = 0$$

and the curve of order  $N + 1$  is

$$(21) \quad c_1^{N_0} c_2^{N_1} + s_1^{N_0} s_2^{N_1} = 0$$

where  $N_3 = \lfloor \frac{N-2}{2} \rfloor$  and  $N_0 = \lfloor \frac{N+1}{2} \rfloor$ .

We can specify points on the curve of order  $N$  that have vertical tangents by having (19) hold and by requiring the real part of (17) to be zero. Eliminating  $c$  from the latter by means of (18), we have

$$(22) \quad \frac{c_1^{N_1-1} c_2^{N_2}}{c_2^{N_2}} - \frac{c_1^{N_1} c_2^{N_2}}{c_1^{N_1}} = \frac{c_2^{N_2} c_1^{N_1-1}}{c_2^{N_2}} - \frac{c_2^{N_2} c_1^{N_1}}{c_1^{N_1}}.$$

It remains to show that a point specified by (19) and (22) lies either on curve (20) or (21).

We consider separately the cases of  $N$  even and odd. For  $N = 2\bar{N}$ , we have  $N_0 = \bar{N}$ ,

$N_1 = \bar{N}$ ,  $N_2 = \bar{N} - 1$ ,  $N_3 = \bar{N} - 1$ . Then (19) and (22) can be written

$$\begin{pmatrix} c_2^{\bar{N}-1} & s_2^{\bar{N}-1} \\ -s_1^{\bar{N}-1} & c_1^{\bar{N}-1} \end{pmatrix} \begin{pmatrix} c_1^{\bar{N}} \\ s_1^{\bar{N}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Either  $c_1^{\bar{N}-1} c_2^{\bar{N}-1} + s_1^{\bar{N}-1} s_2^{\bar{N}-1} = 0$  or the trivial solution  $c_1^{\bar{N}} = 0$ ,  $s_1^{\bar{N}} = 0$  holds. The former is (20) and the latter satisfies (21). Then for  $N = 2\bar{N} + 1$ , we have  $N_0 = \bar{N} + 1$ ,  $N_1 = \bar{N}$ ,  $N_2 = \bar{N}$ ,  $N_3 = \bar{N} - 1$ , and (19) and (22) become

$$\begin{pmatrix} c_1^{\bar{N}} & s_1^{\bar{N}} \\ -s_2^{\bar{N}-1} & c_2^{\bar{N}-1} \end{pmatrix} \begin{pmatrix} c_2^{\bar{N}} \\ s_2^{\bar{N}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Either  $C_1^{\bar{N}} C_2^{\bar{N}-1} + S_1^{\bar{N}} S_2^{\bar{N}-1} = 0$  or  $C_2^{\bar{N}} = 0, S_2^{\bar{N}} = 0$ . The former is (20) and the latter satisfies (21).

The preceding gives some evidence that the chains of curves of Figure 1 extend indefinitely. It is impossible, for example, to have an isolated smooth closed curve that could be missed in the numerical search for such curves, since any point where the tangent is vertical must necessarily lie on another curve as well.



#### 4. A Nonlinear Example

The equation

$$(23) \quad \frac{dy}{dx} = -2xy^2,$$

used by Rosser [4], is chosen to illustrate the accuracy of the method. In anticipation of treating partial differential equations by this method we used the implicit scheme with  $\lambda = \frac{1}{2}$  since this is the most stable form of the difference scheme. For strict economy of effort in solving (23) the explicit scheme with  $\lambda = 0$  would be better since it is stable for small enough  $h$  in the sense that errors do not dominate the solution and it is solved by substitution. The coefficients (3) were derived for equation (23) for  $N = 1$  to 5 and system (5) set up. The first equation of (5),

$$(24) \quad \sum_{n=0}^N \frac{\bar{a}_n(x_0 + \frac{h}{2}, \bar{a}_0) (-\frac{h}{2})^n}{n!} = y(x_0)$$

is a nonlinear equation for  $\bar{a}_0$  and must be solved by iteration. Newton's method was used since the derivatives of  $\bar{a}_n$  with respect to  $\bar{a}_0$  are easily found from formulas (3). Since for large values of  $h$  the solution can change a great deal in the distance  $\frac{h}{2}$  between the point  $x_0$  where the solution is known and the point of expansion, the first guess for Newton's method really should not be the last known solution value but the much more accurate value generated by the explicit method.

Thus the formula

$$(25) \quad \bar{a}_0 \approx \sum_{n=0}^N \frac{a_n(x_0, y(x_0)) (\frac{h}{2})^n}{n!}$$

is used as a "predictor" since it is highly accurate but lacks A-stability. Then a few iterations of Newton's method on (24) with (25) as initial guess "correct" the solution.

The initial value for (23) was taken as  $y(-8) = \frac{1}{65}$  so that the exact solution is  $(1 + x^2)^{-1}$ . The solution and the error grow as  $x$  approaches zero but as  $x$  increases beyond zero both decay. The maximum of the relative error



$$\frac{y_{\text{computed}} - y_{\text{exact}}}{y_{\text{exact}}}$$

was tabulated and usually occurred at  $x = 0$ . There are present in the table some random effects and some effects due to symmetry about  $x = 0$ . Generally, however, one can detect the improvement in accuracy as  $h$  decreases and as  $N$  increases.

h Step-size	N - Order of Method				
	1	2	3	4	5
2	$\infty$	$-6.5 \times 10^{-1}$	$\infty$	2.1	$-2.9 \times 10^{-1}$
1	$\infty$	$-4.4 \times 10^{-1}$	$4.3 \times 10^{-2}_{(1)}$	$-2.6 \times 10^{-2}$	$1.2 \times 10^{-2}$
.5	$\infty$	$-2.0 \times 10^{-1}$	$1.3 \times 10^{-3}_{(.5)}$	$-1.5 \times 10^{-3}$	$-5.8 \times 10^{-5}_{(.5)}$
.25	$2.7 \times 10^{-1}$	$-6.3 \times 10^{-2}$	$-7.9 \times 10^{-5}$	$-7.5 \times 10^{-5}$	$1.8 \times 10^{-6}$
.1	$3.3 \times 10^{-2}$	$-1.1 \times 10^{-2}$	$-6.5 \times 10^{-6}$	$-1.8 \times 10^{-6}$	$-3.6 \times 10^{-9}_{(.3)}$

Table of Maximum Relative Errors

The maximum relative errors over all grid points are tabulated for the solution of  $\frac{dy}{dx} = -2xy^2$ ,  $y(-8) = \frac{1}{65}$  in the range  $(-8, \infty)$  for various step sizes and orders of the method. An error of  $\infty$  indicates that at some step the difference scheme failed to have a real solution. If the maximum error did not occur at  $x = 0$ , that grid point appears in brackets.

#### Concluding Remarks

In summary, the power series method is offered as a way of discretizing partial differential equations with high order of accuracy in all independent variables. This introductory paper demonstrates the discretization procedure on first order nonlinear ordinary differential equations and analyses A-stability of the method. An implicit form of the method turns out to be A-stable to all orders subject to a mild restriction on the step-size. The key equation (16) that determines that restriction arises frequently when dealing with simple partial differential equations. An example is the heat equation that is dealt with in the following paper.

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